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Restricted *T*-Universal Functions

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We prove the existence of a function φ which is holomorphic exactly in the unit disk $\mathbb{D}$ and has universal translates with respect to a prescribed closed set $E \subset \partial\mathbb{D}$ and satisfies $\varphi \in C^{\infty}(\partial\mathbb{D} \setminus E)$. If Q is a subsequence of $\mathbb{N}_0$ with upper density $\bar{d}(Q) = 1$ then the function φ can be constructed such that in addition

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{with} \quad a_n = 0 \quad \text{if} \quad n \notin Q.$$

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1. INTRODUCTION

In 1929 Birkhoff [1] proved the existence of an entire function φ with the most remarkable property that for every entire function f there exists a sequence $\{z_n\}$ in $\mathbb{C}$ with $z_n \to \infty$ such that

$$\varphi(z+z_n) \to f(z) \qquad \text{compactly on} \quad \mathbb{C}.$$

There exists an extensive literature on variants and strengthenings of this result. Also, many other universal properties were investigated; for details we refer to Grosse-Erdmann [4], where a survey of the various universalities and a full bibliography are given.

For a compact set K in the complex plane $\mathbb{C}$ we denote by $A(K)$ the set of all complex valued functions, which are continuous on K and holomorphic in its interior K°. Introducing the uniform norm, $A(K)$ becomes a Banach space. By $\mathscr{M}$ we denote the family of all compact sets which have connected complement.

In our previous paper [11] the concept of the "T-universality" of a function on an open set $\mathscr{O} \subset \mathbb{C}$ was introduced. Here we are interested in the following variant of universality.

Definition. Let $G \subset \mathbb{C}$ be a domain and $\zeta \in \partial G$ (where ∂G is considered as a subset of the extended plane $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$). A function φ is called T_ζ-universal on G (universal under translates with respect to ζ) if it is holomorphic on G and satisfies the following property: For all $K \in \mathscr{M}$ and all $f \in A(K)$ there exist sequences $\{a_n\}$ and $\{b_n\}$ with

$$a_n z + b_n \in G \qquad \text{for all} \quad z \in K \quad \text{and all} \quad n \in \mathbb{N},$$

$$a_n z + b_n \to \zeta \qquad \text{for all} \quad z \in K, \; n \to \infty,$$

$$\varphi(a_n z + b_n) \to f(z) \qquad \text{uniformly on } K, \; n \to \infty.$$

If the function φ is T_ζ-universal on G for all $\zeta \in \partial G$, then φ is shortly called T-universal on G (see [11]).

In terms of the above definition the Birkhoff function is a T-universal entire function. The existence of T-universal functions on simply connected domains $G \neq \mathbb{C}$ has been proved in [7, 8], and in [11] the authors have shown that T-universal functions can have lacunary power series expansions.

It is a natural question whether a holomorphic function φ can exhibit universal properties in the sense that it is T_ζ-universal on G for all ζ of a prescribed subset $E \subset \partial G$ and such that φ is not T_ζ-universal for all ζ on the complementary boundary part $\partial G \setminus E$. If such a phenomenon occurs then we call φ a "restricted T-universal function."

Here we consider the unit disk $\mathbb{D} := \{z: |z| < 1\}$; if a closed set $E \subset \partial\mathbb{D}$, $E \neq \emptyset$ is prescribed then it is proved that there exists a function φ which is T_ζ-universal for all $\zeta \in E$ and is not T_ζ-universal for all $\zeta \in F := \partial\mathbb{D} \setminus E$. In addition the function φ can be chosen such that it has a lacunary power series expansion and satisfies $\varphi \in C^\infty(F)$. (For the details see Theorem 3.) The construction of such a function follows by an inductive process, where a Lemma on lacunary polynomial approximation and the existence of special (unrestricted) T-universal functions on starlike domains are the essential tools.

2. APPROXIMATION BY LACUNARY POLYNOMIALS

We first prove a result on the approximation of functions by lacunary polynomials. For a subsequence Q of $\mathbb{N}_0$ we denote as usual by $\bar{d}(Q)$ its upper and by $\underline{d}(Q)$ its lower density given by

$$\bar{d}(Q) := \overline{\lim_{n\to\infty}} \frac{\nu_Q(n)}{n}, \qquad \underline{d}(Q) := \underline{\lim_{n\to\infty}} \frac{\nu_Q(n)}{n},$$

where $\nu_Q(n)$ is the number of $m \in Q$ with $m \leqslant n$. In the case when $\bar{d}(Q) = \underline{d}(Q)$ we say that Q has density $d(Q) = \bar{d}(Q)$. Moreover the expression

$$d_{\min}(Q) := \lim_{t\to 1-} \left\{ \underline{\lim_{r\to\infty}} \frac{\nu_Q(r) - \nu_Q(rt)}{(1-t)\, r} \right\}$$

is called the minimal density of Q in the sense of Pólya [13].

The following Lemma—which is of interest in itself—will be frequently used in the following sections.

LEMMA. *Suppose that K is a compact set in $\mathcal{M}$ with $0 \in K^\circ$, and let K_0, the component of K containing 0, be starlike with respect to 0. Suppose further that Q is a subsequence of $\mathbb{N}_0$ with upper density $\bar{d}(Q) = 1$. Let f be a function which is holomorphic on K and has near the origin a power series representation of the form*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \quad \textit{with} \quad f_n = 0 \quad \textit{for} \quad n \notin Q.$$

Then, for every $\varepsilon > 0$, there exists a polynomial P of the form

$$P(z) = \sum_{n=0}^{\infty} p_n z^n \qquad \text{with} \quad p_n = 0 \quad \text{for} \quad n \notin Q \tag{1}$$

such that

$$\max_K |f(z) - P(z)| < \varepsilon.$$

Proof. According to the Hahn–Banach theorem, it is sufficient to show that for every bounded linear functional F on $C(K)$ with $F(P) = 0$ for all polynomials P of the form (1) we have $F(f) = 0$. By the Riesz representation theorem, every bounded linear functional on $C(K)$ is given by a Borel measure μ on K. So we have to prove that for every Borel measure μ on K with

$$\int_K \zeta^n \, d\mu(\zeta) = 0 \qquad \text{for} \quad n \in Q \tag{2}$$

we have

$$\int_K f(\zeta) \, d\mu(\zeta) = 0.$$

Let a Borel measure μ on K with (2) be given, and let

$$h(z) := \int_K \frac{d\mu(\zeta)}{\zeta - z} \qquad \text{for} \quad z \in \hat{\mathbb{C}} \setminus K$$

be the Cauchy transform of μ. Then h is holomorphic on $\hat{\mathbb{C}} \setminus K$, and for $|z| > \max_{\zeta \in K} |\zeta|$ we have

$$h(z) = \int_K \left(-\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} \right) d\mu(\zeta) \tag{3}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \left(-\int_K \zeta^n \, d\mu(\zeta) \right) = -\sum_{n \notin Q} \frac{\mu_n}{z^{n+1}},$$

where $\mu_n := \int_K \zeta^n \, d\mu(\zeta)$ for $n \notin Q$. Since $\bar{d}(Q) = 1$ and thus $\underline{d}(\mathbb{N}_0 \setminus Q) = 0$, a result of Pólya [14, p. 737, Satz B] shows that h has a simply connected domain of existence. Therefore, h has a holomorphic extension to $\hat{\mathbb{C}} \setminus K_0$.

By our assumptions, K_0 is starlike with respect to 0, which implies that the region $S := \{s = 1/z : z \in \hat{\mathbb{C}} \setminus K_0\}$ is also starlike with respect to 0. It is well known that the Mittag–Leffler transform [5, p. 75]

$$M_\alpha(s) = \sum_{n \notin Q} \frac{-\mu_n}{\Gamma(1+n/\alpha)} s^n =: -\sum_{n \notin Q} \beta_n(\alpha)\, s^n$$

tends to $\tilde{h}(s) = h(1/s)/s$ compactly on S for $\alpha \to \infty$. Since M_α is entire for all $\alpha > 0$, we have, for the entire function $R_\alpha(z) := M_\alpha(1/z)/z$ in $1/z$,

$$R_\alpha(z) = -\sum_{n \notin Q} \frac{\beta_n(\alpha)}{z^{n+1}} \qquad (|z| > 0)$$

and

$$R_\alpha(z) \to h(z) \qquad (\alpha \to \infty)$$

compactly on $\hat{\mathbb{C}} \setminus K_0$.

Let $\Omega \subset \mathbb{C}$ be an open set containing K and such that f is holomorphic in Ω. Then there exists a contour Γ in $\Omega \setminus K$ such that

$$\operatorname{ind}_\Gamma(\zeta) = \begin{cases} 1, & \zeta \in K \\ 0, & \zeta \notin \Omega \end{cases}$$

(see, for example, [15, Theorem 13.5]). From Cauchy's theorem we obtain

$$f(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-\zeta}\, dw$$

for all $\zeta \in K$ and

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w^{n+1}}\, dw = f_n = 0 \quad \text{for} \quad n \notin Q.$$

Since $\Gamma \subset \Omega \setminus K \subset \Omega \setminus K_0$ we have uniform convergence of $\{R_\alpha\}$ on Γ, which, together with Fubini's theorem, yields

$$\int_K f(\zeta)\, d\mu(\zeta) = \int_K \left[\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-\zeta}\, dw \right] d\mu(\zeta)$$

$$= \frac{1}{2\pi i} \int_\Gamma f(w) \int_K \frac{d\mu(\zeta)}{w-\zeta}\, dw$$

$$= \lim_{\alpha \to \infty} \frac{1}{2\pi i} \int_\Gamma f(w)(-R_\alpha(w))\, dw$$

$$= \lim_{\alpha \to \infty} \sum_{n \notin Q} \beta_n(\alpha) \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w^{n+1}}\, dw = 0.$$

This proves the lemma. ▌

3. *T*-UNIVERSAL FUNCTIONS ON STARLIKE DOMAINS

For a sequence $\{z_n\}$ of complex numbers we denote by $V(\{z_n\})$ the set of all its accumulation points. Using the Lemma of the previous section we can prove the following result.

THEOREM 1. *Let $G \subset \mathbb{C}, G \neq \mathbb{C}$, be a domain which is starlike with respect to the origin and let Q be a subsequence of $\mathbb{N}_0$ having upper density $\bar{d}(Q) = 1$. Suppose furthermore that $H \subset G$ is a domain with $E := \partial H \cap \partial G \neq \emptyset$, that $\{a_n\}$ is a sequence in $\mathbb{C} \setminus \{0\}$ with $a_n \to 0$ for $n \to \infty$ and that $\{b_n\}$ is a sequence in H with $V(\{b_n\}) = E$.*

Then there exists a function φ holomorphic in G with lacunary power series

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n \qquad \text{with} \quad \varphi_n = 0 \quad \text{for} \quad n \notin Q \tag{4}$$

which has the following property: For all $K \in \mathscr{M}$, for all $f \in A(K)$ and for all $\zeta \in E$ there exist subsequences $\{m_k\}$ and $\{n_k\}$ of $\mathbb{N}$ with $\lim_{k \to \infty} b_{n_k} = \zeta$, $a_{m_k} z + b_{n_k} \in H$ for all $k \in \mathbb{N}$ and all $z \in K$, and such that

$$\varphi(a_{m_k} z + b_{n_k}) \to f(z) \qquad \text{uniformly on} \quad K.$$

Proof. (1) We choose a sequence $\{J_n\}_{n \in \mathbb{N}}$ of Jordan domains which are starlike with respect to 0 with $\bar{J}_n \subset J_{n+1} \subset G$ for all n and the property that for an arbitrary compact set $K \subset G$ there exists an $N = N(K)$ such that $K \subset J_N$.

Suppose that $\{\zeta^{(k)}\}_{k \in \mathbb{N}}$ is a sequence of points $\zeta^{(k)} \in E$ which is dense in E. For each $k \in \mathbb{N}$ we select a subsequence $\{z_\nu^{(k)}\}_{\nu \in \mathbb{N}}$ of $\{b_n\}$ with $\lim_{\nu \to \infty} z_\nu^{(k)} = \zeta^{(k)}$ such that for each $\nu \in \mathbb{N}$ the points $z_\nu^{(1)}, \ldots, z_\nu^{(\nu)}$ are pairwise distinct and such that for a subsequence $\{J_{n_\nu}\}$ of $\{J_n\}$ we have $z_\nu^{(k)} \in G_{\nu+1} \setminus \overline{G}_\nu$ for $k = 1, \ldots, \nu$, where $G_\nu := J_{n_\nu}$.

Next we choose radii $r_\nu := \sqrt{|a_{\ell_\nu}|}$ with the property that the closed disks

$$D_{\nu,k} := \{z : |z - z_\nu^{(k)}| \leqslant r_\nu\}$$

are pairwise disjoint for $k = 1, \ldots, \nu$ and that

$$S_\nu := \bigcup_{k=1}^{\nu} D_{\nu,k} \subset G_{\nu+1} \setminus \overline{G_\nu} \qquad \text{and} \qquad S_\nu \subset H.$$

(2) We construct a sequence of polynomials $\{P_\nu\}$ of the form (1). Consider any enumeration $\{\Omega_n\}$ of all polynomials whose coefficients have rational real and imaginary parts. We set $P_0 \equiv 0$ and assume that for a $\nu \in \mathbb{N}$ the polynomials $P_0, \ldots, P_{\nu-1}$ have already been determined. If we apply the lemma to the function

$$F(z) := \begin{cases} P_{\nu-1}(z) & \text{if} \quad z \in \overline{G_\nu} \\ \Omega_\nu\left(\dfrac{1}{a_{\ell_\nu}}(z - z_\nu^{(k)})\right) & \text{if} \quad z \in D_{\nu,k} \end{cases}$$

which is holomorphic on $\overline{G_\nu} \cup \bigcup_{k=1}^{\nu} D_{\nu,k}$ then we can find a polynomial P_ν of the form (1) which simultaneously satisfies

$$\max_{w \in \bar{G}_\nu} |P_\nu(w) - P_{\nu-1}(w)| < \frac{1}{\nu^2} \tag{5}$$

and

$$\max_{w \in D_{\nu,k}} \left| P_\nu(w) - \Omega_\nu\left(\frac{1}{a_{\ell_\nu}}(w - z_\nu^{(k)})\right)\right| < \frac{1}{\nu} \quad \text{for all} \quad k = 1, 2, \ldots, \nu.$$

By induction, we obtain the sequence $\{P_\nu\}$.

Now we consider the function

$$\varphi(w) := \sum_{\nu=1}^{\infty} \{P_\nu(w) - P_{\nu-1}(w)\}.$$

It follows from (5) that φ is holomorphic on G and it obviously has a power series representation of the form (4). Furthermore we obtain, for $k = 1, 2, \ldots, \nu$; $\nu = 1, 2, \ldots$, the estimates

$$\max_{w \in D_{\nu,k}} \left| \varphi(w) - \Omega_\nu \left(\frac{1}{a_{\ell_\nu}} (w - z_\nu^{(k)}) \right) \right|$$
$$\leqslant \max_{w \in D_{\nu,k}} \left| \sum_{\lambda=\nu+1}^{\infty} \{P_\lambda(w) - P_{\lambda-1}(w)\} \right| + \frac{1}{\nu}$$
$$\leqslant \sum_{\lambda=\nu+1}^{\infty} \max_{w \in \bar{G}_\lambda} |P_\lambda(w) - P_{\lambda-1}(w)| + \frac{1}{\nu} < \sum_{\lambda=\nu+1}^{\infty} \frac{1}{\lambda^2} + \frac{1}{\nu} < \frac{2}{\nu}.$$

Consequently, for $k = 1, ..., \nu$; $\nu = 1, ...,$ we have

$$\max_{|z| \leqslant 1/r_\nu} |\varphi(z a_{\ell_\nu} + z_\nu^{(k)}) - \Omega_\nu(z)| < \frac{2}{\nu}. \tag{6}$$

(3) Now let a compact set $K \in \mathscr{M}$, a function $f \in A(K)$ and a boundary point $\zeta \in E$ be given. By Mergelian's theorem [3, p. 92; 12] we find a subsequence $\{\nu_s\}$ of $\mathbb{N}$ such that

$$\max_K |\Omega_{\nu_s}(z) - f(z)| < \frac{1}{s}. \tag{7}$$

There exists an $S = S(K)$ such that

$$K \subset \left\{ z : |z| \leqslant \frac{1}{r_{\nu_s}} \right\} \quad \text{for all} \quad s > S.$$

Hence from (6) and (7) we have

$$\max_K |\varphi(z a_{\ell_{\nu_s}} + z_{\nu_s}^{(k)}) - f(z)| < \frac{1}{s} + \frac{2}{\nu_s}$$

for all $s > S$ and all $k = 1, ..., \nu_s$.

Since the set of points $\{z_{\nu_s}^{(k)} : k = 1, ..., \nu_s;\, s \geqslant S\}$ has ζ as an accumulation point, there exist natural numbers $j_s \in \{1, ..., \nu_s\}$ such that $z_{\nu_s}^{(j_s)} \to \zeta$ for $s \to \infty$. Therefore, we have

$$\max_K |\varphi(z a_{\ell_{\nu_s}} + z_{\nu_s}^{(j_s)}) - f(z)| \to 0 \qquad (s \to \infty),$$

which proves Theorem 1. ∎

Remark 1. In general, the lacunary condition $\bar{d}(Q)=1$ in Theorem 1 cannot be weakened: For the domain $G:=\mathbb{C}\setminus[1,\infty)$, which is obviously starlike with respect to the origin, it follows from Satz A in [14, p. 737], that every function φ which is holomorphic in G and not entire, has a power series representation

$$\varphi(z)=\sum_{n=0}^{\infty}\varphi_n z^n$$

around 0 such that $\varphi_n\neq 0$ for all $n\in Q$, where Q is some subsequence of $\mathbb{N}_0$ with $\bar{d}(Q)=1$, i.e. the nonvanishing coefficients have upper density 1. Theorem 1 shows in particular that such functions exist and that they in addition can satisfy universality conditions with respect to the boundary $[1,\infty)$.

Moreover, we emphasize that in general the subsequences $\{m_k\}$ and $\{n_k\}$ in Theorem 1 cannot be chosen to be identical. If, for example, $G=\mathbb{D}$ and $a_n=1/\sqrt{n}$, $b_n=1-1/n$, then for no compact set K with $1\in K$ the condition $a_{n_k}z+b_{n_k}\in\mathbb{D}$ for all $z\in K$ and infinitely many n_k can be satisfied.

We finally mention that it would be interesting to characterize those simply connected domains G for which the corresponding statements of Theorem 1 hold.

In Theorem 1 we assumed that G is not the entire plane. In the case of $G=\mathbb{C}$ we have

THEOREM 2. *Let Q be any subsequence of $\mathbb{N}_0$ with upper density $\bar{d}(Q)=1$ and let $\{z_n\}$ be any unbounded sequence of complex numbers. Then there exists an entire function φ with lacunary power series*

$$\varphi(z)=\sum_{n=0}^{\infty}\varphi_n z^n \qquad \textit{with} \quad \varphi_n=0 \quad \textit{for} \quad n\notin Q$$

which is universal in the double sense that

- *The sequence of "additive translates" $\{\varphi(z+z_n)\}$ is dense in $A(K)$ for all $K\in\mathcal{M}$.*
- *The sequence of "multiplicative translates" $\{\varphi(z\cdot z_n)\}$ is dense in $A(K)$ for all $K\in\mathcal{M}$ with $0\notin K$.*

Theorem 2 generalizes the main result of [10], where Q is assumed to have density $d(Q)=1$; it is proved by exactly the reasonings as the corresponding particular case with the only change that the Lemma is used instead of Lemma 2 from [10].

4. RESTRICTED T-UNIVERSAL FUNCTIONS IN THE UNIT DISK

The following Theorem is our main result.

THEOREM 3. *Let $E \subset \partial\mathbb{D}$ be a closed set and define $F := \partial\mathbb{D} \setminus E$. Let Q be a subsequence of $\mathbb{N}_0$ with upper density $\bar{d}(Q) = 1$ and suppose that $\{b_n\}$ is a sequence in $\mathbb{D}$ with $V(\{b_n\}) = E$ and that $\{a_n\}$ is a sequence in $\mathbb{C} \setminus \{0\}$ with $a_n \to 0$ for $n \to \infty$.*

Then there exists a function φ which is holomorphic exactly on $\mathbb{D}$ and has a lacunary power series

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n \qquad \text{with} \quad \varphi_n = 0 \quad \text{for} \quad n \notin Q$$

such that φ belongs to $C^\infty(F)$ and satisfies: For all $K \in \mathscr{M}$, for all $f \in A(K)$ and for all $\zeta \in E$ there exist subsequences $\{m_k\}$ and $\{n_k\}$ of $\mathbb{N}$ with $a_{m_k} z + b_{n_k} \in \mathbb{D}$ for all $k \in \mathbb{N}$ and all $z \in K$, $\lim_{k\to\infty} b_{n_k} = \zeta$ and such that

$$\varphi(a_{m_k} z + b_{n_k}) \to f(z) \quad \text{uniformly on} \quad K.$$

In particular, the function φ is T_ζ-universal for all $\zeta \in E$ and is not T_ζ-universal for all $\zeta \in F$.

Proof. We consider a subsequence $Q_0 \subset \mathbb{N}$ of Q with density $d(Q_0) = 0$ and define the function ψ by

$$\psi(z) := \sum_{\nu \in Q_0} \frac{z^\nu}{\nu^{\log \nu}} \qquad (z \in \overline{\mathbb{D}}).$$

Then, by Fabry's gap theorem [5, p. 89; 6, pp. 83, 168], ψ is holomorphic exactly on $\mathbb{D}$ and it obviously satisfies $\psi \in C^\infty(\partial\mathbb{D})$. We choose a domain G which is starlike with respect to the origin and satisfies $\mathbb{D} \cup F \subset G$, $E \subset \partial G$.

Let φ_0 be a universal function which satisfies the properties of Theorem 1 for the domains G and $H = \mathbb{D}$. It will be shown that the function

$$\varphi(z) := \varphi_0(z) + \psi(z)$$

has all the desired properties. Obviously φ is holomorphic exactly on $\mathbb{D}$ and in addition we have $\varphi \in C^\infty(F)$, so that φ cannot be T_ζ-universal for any $\zeta \in F$.

Let be given a compact set $K \in \mathscr{M}$, a function $f \in A(K)$ and a boundary point $\zeta \in E$. By Theorem 1, there exist subsequences $\{m_k\}$ and $\{n_k\}$ of $\mathbb{N}$ such that (according to the properties of the function φ_0) the following conditions hold:

$$a_{m_k} z + b_{n_k} \in \mathbb{D} \quad \text{for all} \quad z \in K \quad \text{and all} \quad k \in \mathbb{N}, \qquad \lim_{k\to\infty} b_{n_k} = \zeta$$

and

$$\varphi_0(a_{m_k}z+b_{n_k}) \to f(z)-\psi(\zeta) \qquad \text{uniformly on} \quad K.$$

Hence we have

$$\varphi(a_{m_k}z+b_{n_k}) \to f(z) \qquad \text{uniformly on} \quad K,$$

which proves the result. ▮

Remark 2. In general the condition $\bar{d}(Q)=1$ in Theorem 3 cannot be weakened in the sense that for every (rational) number $d \in (0, 1)$ a sequence Q in $\mathbb{N}_0$ exists with $d(Q)=d$ and such that for every function φ with lacunary power series (4) a universality property for $E=\{1\}$ and a non-universality property for $F=\partial\mathbb{D}\setminus\{1\}$ as in Theorem 3 cannot hold simultaneously.

In the case $d=1/q$, where $q \in \mathbb{N}$, this is easily seen: For $Q=q\mathbb{N}_0$ we have $d(Q)=d$. If φ is holomorphic in $\mathbb{D}$ with (4), then φ is a holomorphic function of z^q in $\mathbb{D}$. The condition $\varphi \in C^\infty(\partial\mathbb{D}\setminus\{1\})$ obviously implies $\varphi \in C^\infty(\partial\mathbb{D})$ due to the symmetry conditions satisfied by φ.

Now we consider the case $d=2/3$ setting $Q=\{0, 1, 3, 4, 6, 7, \dots\}$. For φ holomorphic in $\mathbb{D}$ with (4) we have

$$\varphi(z)=\varphi_1(z^3)+z\varphi_2(z^3) \tag{8}$$

with φ_1, φ_2 being holomorphic in $\mathbb{D}$. If $w:=e^{2\pi i/3}$ then

$$w^2\varphi_1(z^3)=w^2\varphi_1((wz)^3)=w^2\varphi(wz)-z\varphi_2(z^3)$$

and thus

$$(1-w^2)\,\varphi_1(z^3)=\varphi(z)-w^2\varphi(wz).$$

Therefore, $\varphi \in C^\infty(\partial\mathbb{D}\setminus\{1\})$ implies that $\psi_1 \in C^\infty(\partial\mathbb{D}\setminus\{1, w^{-1}\})$, where $\psi_j(z)=\varphi_j(z^3)$ for $j=1, 2$, which again implies $\psi_1 \in C^\infty(\partial\mathbb{D})$ by symmetry. From (8) it is now easily seen that also $\psi_2 \in C^\infty(\partial\mathbb{D})$, and thus $\varphi \in C^\infty(\partial\mathbb{D})$.

Similar considerations lead to examples of sequences Q for every $d=p/q$, where $p, q \in \mathbb{N}$, $p<q$ (cf. [2]).

The situation changes drastically if we no longer require nonuniversality on some parts of $\partial\mathbb{D}$ (as in the case $E=\partial\mathbb{D}$ in Theorem 3). In this case, we have following result.

THEOREM 4. *Let Q be a subsequence in $\mathbb{N}_0$ with minimal density $d_{\min}(Q)>0$, $\{a_n\}$ be a sequence in $\mathbb{C}\setminus\{0\}$ with $a_n \to 0$ for $n \to \infty$ and $\{b_n\}$ be* *a sequence in* $\mathbb{D}$ *with* $V(\{b_n\}) = E \subset \partial\mathbb{D}$. *Then there exists a function* φ *holomorphic in* $\mathbb{D}$ *with lacunary power series*

$$\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n \qquad \text{with} \quad \varphi_n = 0 \quad \text{for} \quad n \notin Q$$

which has the following property: For all $K \in \mathcal{M}$, *for all* $f \in A(K)$ *and for all* $\zeta \in E$ *there exist subsequences* $\{m_k\}$ *and* $\{n_k\}$ *of* $\mathbb{N}$ *with* $a_{m_k} z + b_{n_k} \in \mathbb{D}$ *for all* $k \in \mathbb{N}$ *and all* $z \in K$, $\lim_{k\to\infty} b_{n_k} = \zeta$ *and such that*

$$\varphi(a_{m_k} z + b_{n_k}) \to f(z) \qquad \text{uniformly on} \quad K.$$

The particular case of Theorem 4 when $E = \partial\mathbb{D}$ and where the sequences $\{a_n\}$, $\{b_n\}$ are not preassigned, is proved in [11, Theorem 2]. Theorem 4 is proved essentially in the same way using the construction in the beginning of the proof of Theorem 1 above.

Remark. In a forthcoming article the authors will investigate a similar question as in Theorem 3 for more general domains.

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